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2006 IMO Shortlist A5

Let a, b, c be the sides of a triangle. Prove that

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}+\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3.$$

WLOG, $a \geq b \geq c$ and $a+b+c=1$

lemma. $\dfrac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} \leq 1$

pf) It suffices to show that

$$\sqrt{b+c-a} \leq \sqrt{b}+\sqrt{c}-\sqrt{a}$$

$$\sqrt{1-2a} \leq \sqrt{b}+\sqrt{c}-\sqrt{a}$$

$$\sqrt{a}+\sqrt{1-2a} \leq \sqrt{b}+\sqrt{c}$$

$$a+1-2a+2\sqrt{a(1-2a)} \leq b+c+2\sqrt{bc}$$

$$\cancel{1}+2\sqrt{a(1-2a)} \leq \cancel{a+b+c}+2\sqrt{bc}$$

$$a(1-2a) \leq bc$$

$$a(b+c-a) \leq bc$$

$$0 \leq a^2+bc-ab-ca = \underbrace{(a-b)}_{+}\underbrace{(a-c)}_{+}$$

$\square$

$$0 \leq b^2+ca-bc-ab$$

$$\underbrace{(b-c)}_{+}\underbrace{(b-a)}_{-} \leq 0$$

Lemma 2. $\dfrac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}+\dfrac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 2$

let $p=\sqrt{b}+\sqrt{c}$, $q=\sqrt{b}-\sqrt{c}$, $b-c=pq$

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$$\frac{\sqrt{a-pq}}{\sqrt{a}-q}+\frac{\sqrt{a+pq}}{\sqrt{a}+q}\le 2$$

$$(\alpha x+\beta y)^2\le(\alpha^2+\beta^2)(x^2+y^2)$$

$$\alpha=\frac{\sqrt{a-pq}}{\sqrt{\sqrt{a}-q}},\quad x=\frac{1}{\sqrt{\sqrt{a}-q}}$$

$$\left(\frac{a-pq}{\sqrt{a}-q}+\frac{a+pq}{\sqrt{a}+q}\right)\left(\frac{1}{\sqrt{a}-q}+\frac{1}{\sqrt{a}+q}\right)\le 4$$

$$\frac{a\sqrt{a}+\cancel{aq}-\cancel{pq\sqrt{a}}-pq^2+a\sqrt{a}+\cancel{pq\sqrt{a}}-\cancel{aq}-pq^2}{a-q^2}\cdot\frac{2\sqrt{a}}{a-q^2}\le 4$$

$$\frac{2a\sqrt{a}-2pq^2}{a-q^2}\cdot\frac{2\sqrt{a}}{a-q^2}\le 4$$

$$4\cdot\frac{\sqrt{a}\,(a\sqrt{a}-2pq^2)}{(a-q^2)^2}\le 4$$

$$\frac{a^2-2pq^2\sqrt{a}}{(a-q^2)^2}\le 1$$

$$a^2-2pq^2\sqrt{a}\le a^2-2aq^2\le a^2-2aq^2+q^4$$

$$\cancel{-2}\,p\cancel{q^2}\sqrt{a}\le\cancel{-2}\,a\cancel{q^2}$$

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$$\sqrt{b}\sqrt{a} \geq a$$

$$\sqrt{b} \geq \sqrt{a}$$

$$\sqrt{b} + \sqrt{c} \geq \sqrt{a}$$

$$b + c + 2\sqrt{bc} \geq a \qquad \because \underline{b + c > a}$$

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By lemma 1 and 2,

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3$$

$\square$