Enoch Yu

For arbitrary real numbers $x_1, x_2, \ldots, x_{99}$ such that $0 < x_1 < x_2 < \cdots < x_{99}$, determine the minimum value of positive real number c such that the following inequality always holds.

$$\underline{3}\sqrt{x_1} + \underline{4}\left(\sqrt{x_2 - x_1} + \sqrt{x_3 - x_2} + \cdots + \sqrt{x_{99} - x_{98}}\right) \le c\left(\sqrt{x_2} + \sqrt{x_3} + \cdots + \sqrt{x_{98}}\right) + \underline{5}\sqrt{x_{99}}$$

$$(a^2+b^2)(x^2+y^2) \geq (ax+by)^2$$

$$(3^2+4^2)(x_1+x_2-x_1) \geq (3\sqrt{x_1} + 4\sqrt{x_2-x_1})^2$$

$$3\sqrt{x_1} + 4\sqrt{x_2-x_1} \leq 5\sqrt{x_2}$$

$a : x = b : y$

$$3\sqrt{x_2} + 4\sqrt{x_3-x_2} \leq 5\sqrt{x_3}$$

$\vdots$

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$$3\sqrt{x_{98}} + 4\sqrt{x_{99}-x_{98}} \leq 5\sqrt{x_{99}}$$

$$3(\sqrt{x_1} + \cdots + \sqrt{x_{98}}) + 4(\sqrt{x_2-x_1} + \cdots + \sqrt{x_{99}-x_{98}}) \leq 5(\sqrt{x_2} + \cdots + \sqrt{x_{99}})$$

$$3\sqrt{x_1} + 4(\sqrt{x_2-x_1} + \cdots + \sqrt{x_{99}-x_{98}}) \leq 2(\sqrt{x_2} + \cdots + \sqrt{x_{98}}) + 5\sqrt{x_{99}}$$

$\therefore$ It suffices to show that $c \geq 2$.

$\Leftarrow$

Inequality above are strict.

$$(a^2+b^2)(x^2+y^2) \geq (ax+by)^2$$

$$a^2y^2 + b^2x^2 \geq 2axby$$

$$(ay-bx)^2 \geq 0$$