Enoch Yu

Let a, b, c be positive real numbers with $abc = 1$. Show that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1$$

$a = \frac{x}{y}, \ b = \frac{y}{z}, \ c = \frac{z}{x} \qquad x, y, z \in \mathbb{R}^+$

$\left(\frac{x}{y} - 1 + \frac{z}{y}\right)\left(\frac{y}{z} - 1 + \frac{x}{z}\right)\left(\frac{z}{x} - 1 + \frac{y}{x}\right) \leq 1$

$\underbrace{(x - y + z)}_{\alpha}\underbrace{(y - z + x)}_{\beta}\underbrace{(z - x + y)}_{\gamma} \leq xyz$

$\alpha\beta\gamma \leq \frac{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}{8}$

$8\alpha\beta\gamma \leq (\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$

i) $\alpha, \beta, \gamma \geq 0$

By AM-GM, true!!

ii) WLOG, $\alpha < 0$, $\beta, \gamma \geq 0$

$8\alpha\beta\gamma \leq 0$, $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) \geq 0$

iii) WLOG, $\alpha, \beta < 0$, $\gamma \geq 0$

$x + z < y$

$x + y < z$

$2x < 0$

$\square$