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Find the minimum of a real number M that satisfies the following inequality for positive real numbers $a_1, a_2, \ldots, a_{99}$ where $a_{100} = a_1$ and $a_{101} = a_2$.

$$\sum_{k=1}^{99} \frac{a_{k+1}}{a_k + a_{k+1} + a_{k+2}} < M$$

$$\frac{a_{k+1}}{a_k + a_{k+1} + a_{k+2}} + \frac{a_{k+2}}{a_{k+1} + a_{k+2} + a_{k+3}} < \frac{a_{k+1}}{a_{k+1} + a_{k+2}} + \frac{a_{k+2}}{a_{k+1} + a_{k+2}} = 1$$

lemma $\exists\, k$ such that $\frac{a_{k+1}}{a_k + a_{k+1} + a_{k+2}} + \frac{a_{k+2}}{a_{k+1} + a_{k+2} + a_{k+3}} + \frac{a_{k+3}}{a_{k+2} + a_{k+3} + a_{k+4}} \le 1$

Proof $\frac{\textcircled{1}}{\textcircled{0} + \textcircled{1} + \textcircled{2}} + \frac{\textcircled{2}}{\textcircled{1} + \textcircled{2} + \textcircled{3}} + \frac{\textcircled{3}}{\textcircled{2} + \textcircled{3} + \textcircled{4}} \le \frac{\textcircled{1}}{\textcircled{3} + \textcircled{1} + \textcircled{2}} + \frac{\textcircled{2}}{\textcircled{3} + \textcircled{1} + \textcircled{2}} + \frac{\textcircled{3}}{\textcircled{1} + \textcircled{2} + \textcircled{3}}$

$\textcircled{0} \geq \textcircled{3} \quad \wedge \quad \textcircled{4} \geq \textcircled{1}$

$a_k \geq a_{k+3} \quad \wedge \quad a_{k+4} \geq a_{k+1}$

FTSOC, $a_k < a_{k+3} \quad \wedge \quad a_{k+4} < a_{k+1} \quad \forall\, k \in \{1, \ldots, 99\}$

$a_1 < a_4$ | $a_5 < a_2$
$a_2 < a_5$ | $a_6 < a_3$
$a_3 < a_6$ | $a_7 < a_4$
$\vdots$ | $\vdots$
$a_{99} < a_1$ | $a_2 < a_{98}$

$$\sum_{k=1}^{99} \frac{\textcircled{1}}{\textcircled{0} + \textcircled{1} + \textcircled{2}} < 49$$

$a_1 = a_3 = \cdots = a_{99} = 1$

$a_2 = a_4 = \cdots = a_{98} = 0$

$$\left(\frac{0}{1+0+1} + \frac{1}{0+1+0}\right) + ()\cdots() \cdots + \frac{0}{1+0+1} + \frac{1}{0+1+1} + \frac{1}{1+1+0}$$

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$$= 48 + \frac{1}{2} + \frac{1}{2} = 49$$

$$a_1 = a_3 = \cdots = a_{49} = 1$$

$$a_2 = a_4 = \cdots = a_{48} = x$$

$$S = \left(\frac{x}{1+x+1} + \frac{1}{x+1+x}\right) + (\;)\cdots(\;) \cdots + \frac{x}{1+x+1} + \frac{1}{x+1+1} + \frac{1}{1+1+x}$$

$$\lim_{x \to 0} S = 49$$

$$M = 49$$

$$49 \leq M \leq 49$$

$$\therefore M = 49$$

$\square$