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Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where $\lfloor a \rfloor$ is greatest integer not greater than a.

$x = 0$

$f(0) = f(0)\lfloor f(y) \rfloor \Rightarrow \underline{f(0) = 0}$ or $\lfloor f(y) \rfloor = 1 \quad \forall y \in \mathbb{R}$

$y = 0$

$f(0) = f(x)\lfloor f(0) \rfloor \Rightarrow \underline{f(0) = 0}$ or $f(x)$ is a constant

i) $f(0) \neq 0$

$\lfloor f(y) \rfloor = 1 \quad \forall y \in \mathbb{R}$

$\underline{f(x) = c, \quad 1 \leq c < 2}$ $\qquad c = c\lfloor c \rfloor$

ii) $f(0) = 0$

1. Claim: $\exists \alpha \in [0,1)$ s.t. $f(\alpha) \neq 0$

$x = \alpha \quad f(0) = f(\alpha)\lfloor f(y) \rfloor \quad \forall y \in \mathbb{R}$

$\lfloor f(y) \rfloor = 0 \quad \forall y \in \mathbb{R}$

$0 \leq f(y) < 1 \quad \forall y \in \mathbb{R}$

$f(y) = f(1)\lfloor f(y) \rfloor \quad \forall y \in \mathbb{R}$

$= 0$

Contradiction!!

2. Claim: $\forall \alpha \in [0,1)$, $\quad f(\alpha) = 0$

$\alpha = \frac{a}{b}$ [illegible]

$\alpha = \frac{a}{n}$ for some $a \in \mathbb{R}$, $n \in \mathbb{N}$.

$$f(a) = f(\alpha n) = f(\lfloor n \rfloor \alpha) = f(n)\left[f(\alpha)\right] \geq 0$$

$$f(a) \geq 0 \quad \forall\, a \in \mathbb{R}$$

$$\boxed{f(x) = c \quad \text{for} \quad c \in [1,2) \quad \text{or} \quad f(0) = 0,}$$

$\square$