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If $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ are all polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x),$$

prove that $x - 1$ is a factor of $P(x)$.

$\Leftarrow$

$P(1)=0$

① $\qquad x^5-1=0$

$w_1, \ldots, w_5 \Rightarrow 5^{th}$ roots of unity where $w_5=1$.

$$P(1)+w_1 Q(1)+w_1^2 R(1)=0$$
$$P(1)+w_2 Q(1)+w_2^2 R(1)=0$$
$$P(1)+w_3 Q(1)+w_3^2 R(1)=0$$
$$P(1)+w_4 Q(1)+w_4^2 R(1)=0$$

②

$$w_1 P(1)+w_1^2 Q(1)+w_1^3 R(1)=0$$
$$w_2 P(1)+w_2^2 Q(1)+w_2^3 R(1)=0$$
$$w_3 P(1)+w_3^2 Q(1)+w_3^3 R(1)=0$$
$$w_4 P(1)+w_4^2 Q(1)+w_4^3 R(1)=0$$

$$4P(1)-P(1)=0$$

$$\therefore P(1)=0$$

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