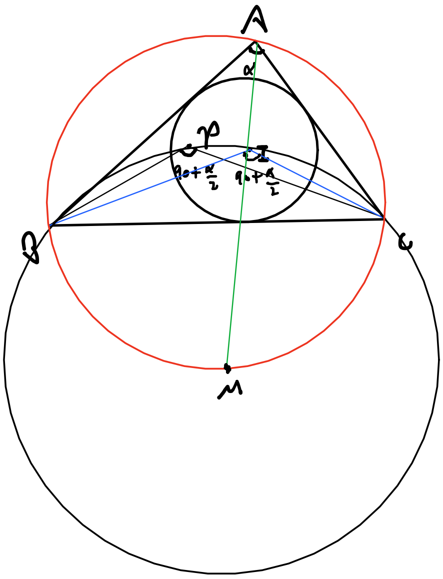
Enoch Yu

Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$ and that equality hold if and only if P coincides with I.



$\angle A = \alpha$

$\angle B = \beta$

$\angle C = \gamma$

$\angle PBA + \angle PBC + \angle PCB + \angle ACP$

$= 180 - \alpha$

$2(\angle PBA + \angle PCA) = 2(\angle PBC + \angle PCB)$

$= 180 - \alpha$

$\angle PBC + \angle PCB = 90 - \frac{\alpha}{2}$

$\angle BPC = 180 - (\angle PBC + \angle PCB)$

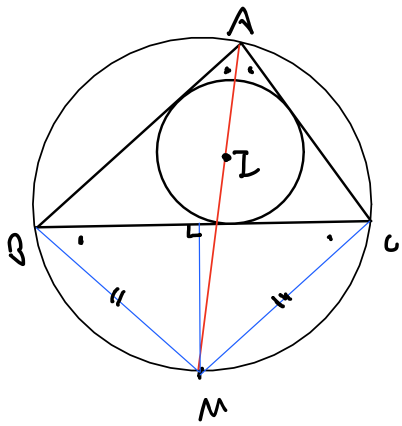
$= 90 + \frac{\alpha}{2}$

(Equality at $P \equiv I$)

$AP + PM \geq AI + IM = AI + PM$

$\therefore AP \geq AI$ (equality iff $P \equiv I$). □

Lemma 1.



Lemma 2.