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Suppose a, b, c are real numbers such that $a+b+c=0$ and $a^2+b^2+c^2=1$. Prove that

$$a^2b^2c^2 \le \frac{1}{54},$$

and determine the cases of equality.

$c = -(a+b)$ ①

$a^2+b^2+c^2 = a^2+b^2+a^2+2ab+b^2 = 2(a^2+ab+b^2) = 1$

$\therefore a^2+ab+b^2 = \frac{1}{2}$ ②

$a^2+ab+b^2 \ge ab+2ab = 3ab$

$a^2+ab+b^2 \ge 3\sqrt[3]{a^3b^3} = 3ab$ } assuming that $a,b \ge 0$

$ab \le \frac{1}{6}$ (equality when $a=b$)

$a^2b^2 \le \frac{1}{36}$ ③

$c^2 = a^2+2ab+b^2 = \frac{1}{2}+ab \le \frac{1}{2}+\frac{1}{6} = \frac{2}{3}$ (equality at $a=b$)

$a^2b^2c^2 \le \frac{1}{54}$

$a = \frac{1}{\sqrt{6}}, \; b = \frac{1}{\sqrt{6}}, \; c = -\frac{2}{\sqrt{6}}$

$\frac{1}{\cancel{6}_3} \cdot \frac{1}{\cancel{6}_3} \cdot \frac{\cancel{4}}{6} = \frac{1}{54}$

$\square$

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i) WLOG, $a,b > 0$ and $c < 0$

we proved it

ii) WLOG, $a,b < 0$, $c > 0$

$a', b' > 0$, $c' < 0$

$a = -a'$ $a+b+c=0$

$b = -b'$

$c = -c'$

$$a^2 + b^2 + c^2 = 1$$

$$a'^2 b'^2 c'^2 \le \frac{1}{54}$$

$$(-a')^2 (-b')^2 (-c')^2 \le \frac{1}{54}$$

$$a^2 b^2 c^2 \le \frac{1}{54}$$

iii) WLOG, $a = 0$, $b > 0$, $c < 0$

$b = -c$

$b = \frac{1}{\sqrt{2}}$