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Let $a_1, a_2, \ldots, a_{2023}$ be positive real numbers with

$$a_1 + a_2^2 + a_3^3 + \cdots + a_{2023}^{2023} = 2023.$$

Show that

$$a_1^{2023} + a_2^{2022} + \cdots + a_{2022}^2 + a_{2023} > 1 + \frac{1}{2023}.$$

$n \geq 5$

Prove:

If
$a_1^{2023} + a_2^{2022} + \cdots + a_{2023} \leq 1 + \frac{1}{2023}$,
then
$a_1 + a_2^2 + \cdots + a_{2023}^{2023} \neq 2023$.

If
$a_1^{2023} + a_2^{2022} + \cdots + a_{2023} \leq 1 + \frac{1}{2023}$,
then
$a_1 + a_2^2 + \cdots + a_{2023}^{2023} < 2023$.

pf)

If $a_i < 1$, then, the statement is true,
Moreover, if more than one a_i are greater than or equal to one, the assumption is false, Therefore, it suffices to prove the case when one $a_i \geq 1$.

$1 \leq a_i < 1 + \frac{1}{2023}$

$$a_i^i < \left(1 + \frac{1}{2023}\right)^i \leq \left(1 + \frac{1}{2023}\right)^{2023} = \sum_{k=0}^{2023} \binom{2023}{k} 1^{2023-k} \cdot \frac{1}{2023^k}$$

$$= \sum_{k=0}^{2023} \binom{2023}{k} \frac{1}{2023^k}$$

$$= \sum_{k=0}^{2023} \frac{2023!}{k!(2023-k)!} \cdot \frac{1}{2023^k}$$

$$= 1 + \sum_{k=1}^{2023} \frac{1}{k!} \cdot \frac{2023}{2023} \cdots \frac{2023-k+1}{2023}$$

$$< 1 + \sum_{k=1}^{2023} \frac{1}{k!}$$

$$< 1 + \sum_{k=0}^{2022} \frac{1}{2^k}$$

$$\frac{1}{1} + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2 \cdot 1} + \cdots + \frac{1}{2023!}$$

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 2} + \cdots + \frac{1}{2^{2022}}$$

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$$= 1 + \frac{1\left(1-\left(\frac{1}{2}\right)^{2021}\right)}{1-\frac{1}{2}}$$

$$= 1 + 2 - \left(\frac{1}{2}\right)^{2021} < 3 \qquad (2021 \to n)$$

$$\therefore a_i^i < 3$$

$$\sum_{k=1,\, k\neq i}^{1011} a_k^k < 1011$$

(1011 → $\frac{n}{2}-1$ or $\frac{n-1}{2}$; 1011 → $\frac{n}{2}$ or $\frac{n-1}{2}$)

$$\sum_{k=1012,\, k\neq i}^{2023} a_k^k < \sum_{k=1012,\, k\neq i}^{2023} a_k^{2024-k} < \sum_{k=1}^{2023} a_k^{2024-k} - a_i^i \le 1 + \frac{1}{2023} - a_i^i$$

$$\le \frac{1}{2023}$$

(2023 → n; 1012 → $\frac{n}{2}$ or $\frac{n+1}{2}$; 2024 → $n+1$)

$$a_{1012}^{1012} + a_{1013}^{1013} + \dots + a_{2023}^{a_{2023}}$$

$$a_{1012}^{1012} + a_{1013}^{1011} + \dots + a_{2023}^{1}$$

$$\therefore \sum_{k=1012,\, k\neq i}^{2023} a_k^k < \frac{1}{2023}$$

(2023 → n; 1012 → $\frac{n}{2}$ or $\frac{n+1}{2}$)

$$\sum_{k=1,\, k\neq i}^{1011} a_k^k + \sum_{k=1011,\, k\neq i}^{2023} a_k^k + a_i^i < 1011 + \frac{1}{2023} + 3 < 2023$$

$$a_1 + a_2^2 + \dots + a_{2023}^{2023} < 2023$$

$$\sum_{k=1,\, k\neq i}^{\frac{n}{2}-1} a_k^k + \sum_{k=\frac{n}{2},\, k\neq i}^{n} a_k^k + a_i^i < \frac{n}{2} - 1 + \frac{1}{n} + 3$$

$$= \frac{n}{2} + \frac{1}{n} + 2$$

$$< n$$

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$$\frac{n-1}{2} + \frac{1}{n} + 2 < \frac{n}{2} + \frac{1}{n} + 2 < n$$

$$\frac{1}{n} + 2 < \frac{n}{2}$$

$$2 + 4n < n^2$$

$$n^2 - 4n - 2 > 0$$

$$2 \pm \sqrt{6}$$

$$n \geq 5$$

$$\frac{n-1}{2} - 1 \qquad n$$

$$+ \qquad \frac{n-1}{2}$$

$$< \frac{n-1}{2} - 1 + \frac{1}{n} + 3$$

$$< n$$