Enoch Yu

Find all functions $f : (0, \infty) \mapsto (0, \infty)$ such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z, satisfying $wx = yz$.

$$\frac{2f(1)^2}{2f(1)} = 1$$

$$f(1)^2 = f(1)$$

$$\therefore f(1) = 1$$

$$\frac{f(2)^2 + 1}{f(2^2) + 1} = 1$$

$$f(2)^2 = f(2^2)$$

$$\frac{f(n)^2 + 1}{f(n^2) + 1} = 1$$

$$\therefore f(n)^2 = f(n^2)$$

$$\frac{f(4)^2 + 1}{2f(4)} = \frac{17}{8}$$

$$34f(4) = 8f(4)^2 + 8$$

$$4f(4)^2 - 17f(4) + 4 = 0$$

$$(4f(4) - 1)(f(4) - 4) = 0$$

$$\therefore f(4) = \frac{1}{4}, 4$$

$$f(x) = x, \quad f(x) = \frac{1}{x}$$

New APPROACH

(w, x, y, z)

$(\sqrt{x}, \sqrt{x}, 1, x)$

1

$$\frac{f(x) + f(x)}{1 + f(x)^2} = \frac{x + x}{1 + x^2}$$

$$2x + 2xf(x)^2 = 2f(x) + 2x^2 f(x)$$

$$2xf(x)^2 - 2(1 + x^2)f(x) + 2x = 0$$

$$(f(x) - x)(xf(x) - 1) = 0$$

FTSOC, let $f(a) \neq a$ and $f(b) \neq \frac{1}{b}$ for some $a, b \neq 1$.

2

$(\sqrt{a}, \sqrt{b}, 1, \sqrt{ab})$

$$\frac{f(a) + f(b)}{1 + f(ab)} = \frac{a + b}{1 + ab}$$

$$\frac{\frac{1}{a} + b}{1 + f(ab)} = \frac{a + b}{1 + ab}$$

i) If $f(ab) = ab$

$\therefore f(x) = x$ or $f(x) = \frac{1}{x}$

$\frac{1}{a} + b = a + b$

$\therefore a = 1$. ×

ii) If $f(a) = \frac{1}{a}$

$$\frac{\frac{1}{a} + b}{1 + \frac{1}{ab}} = \frac{a+b}{1+ab}$$

$b + a^2 b = a + b$

$ab = 1$ ×

$\therefore \boxed{f(x) = x \text{ or } f(x) = \frac{1}{x}}$

$\square$