Enoch Yu

Let $\mathbb{Z}$ be the set of integers. Determine all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that, for all integers a and b,

$$f(2a)+2f(b)=f(f(a+b)).$$

$a=0$

$f(0)+2f(b)=f(f(b))$ (1)

$b=0$

$f(2a)+2f(0)=f(f(a))$ (2)

$= f(0)+2f(a)$

$\therefore f(2a)=2f(a)-f(0)$

1

$f(2a)+2f(b)=f(0)+2f(a+b)$

2

$2f(a)-f(0)+2f(b)=f(0)+2f(a+b)$

$2f(a)-2f(0)+2f(b)-2f(0)=2f(a+b)-2f(0)$

let $g(x)=f(x)-f(0)$ $(g:\mathbb{Z}\to\mathbb{Z})$

$g(a)+g(b)=g(a+b)$

$\therefore f(x)$ is linear

$f(x)=ca$

$2ac+2bc=c(c(a+b))$

$2a+2b=c(a+b)$

3

$c=\frac{2(a+b)}{a+b}$ → trivial solution

$c=2$, $c=0$

$f(x)=2x+k$

$4a+k+4b+2k=f(2a+2b+k)$

$=4a+4b+2k+k$

$$\boxed{\begin{array}{l} f(x)=2x+k \quad (k\in\mathbb{Z}) \\ \text{and} \\ f(x)=0 \quad \text{are the} \\ \text{solutions.}\end{array}}$$

$f(x)=cx+k$

$2ac+\cancel{k}+2bc+2k=f(ac+bc+k)=ac^2+bc^2+ck+\cancel{k}$

$ac(c-2)+bc(c-2)+k(c-2)=0$

$(c-2)(ac+bc+k)=0$

$c=2$ or $c=k=0$

$g(1)=g(0)+g(1)$

$g(2)=g(1)+g(1)$

$g(3)=g(1)+g(2)$

$=g(1)+g(1)+g(1)$

$\vdots$

$g(n)=g(1)+g(n-1)$

$=g(1)+g(1)+g(n-2)$

$\vdots$

$=g(1)+\underbrace{g(1)+\cdots+g(1)}_{n-1}$

$\therefore$