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Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers x and y.

let $x = y = 0$

$f(0) = 0$

$f(x^2 - y^2) = xf(x) - yf(y)$

$f(y^2 - x^2) = yf(y) - xf(x)$

$f(x^2 - y^2) = -f(-x^2 + y^2)$

$\therefore f(x)$ is odd.

let $y = 0$

$f(x^2) = xf(x)$

$f(x^2 - y^2) = f(x^2) - f(y^2)$

let $a = x^2,\ b = -y^2$

$f(a+b) = f(a) + f(b)$

$\therefore \boxed{f(x) = cx \text{ for constant } c.}$

Solve 1) let $a = 2t,\ b = -t$

$f(t) = f(2t) + f(-t)$

$2f(t) = f(2t)$

let $x = t+1,\ y = t$

$f(2t+1) = (t+1)f(t+1) - tf(t)$

$= (t+1)(f(t) + f(1)) - tf(t)$

$= f(t) + f(1) + tf(1)$

$f(2t) + \cancel{f(1)} = f(t) + \cancel{f(1)} + tf(1)$

$2f(t) = f(t) + tf(1)$

$f(t) = tf(1)$

$\therefore \boxed{f(t) = cx \text{ for } c \in \mathbb{R}}$

Solve 2) $f(x) = f\left(\left(\frac{x+1}{2}\right)^2 - \left(\frac{x-1}{2}\right)^2\right)$

$= f\left(\left(\frac{x+1}{2}\right)^2\right) + f\left(-\left(\frac{x-1}{2}\right)^2\right)$

$= f\left(\left(\frac{x+1}{2}\right)^2\right) - f\left(\left(\frac{x-1}{2}\right)^2\right)$

$= \cancel{4} \cdot \frac{1}{\cancel{4}} \cdot f\left(\left(\frac{x+1}{2}\right)^2\right) - 4 \cdot \frac{1}{4} f\left(\left(\frac{x-1}{2}\right)^2\right)$

$= \frac{f((x+1)^2)}{4} - \frac{f((x-1)^2)}{4}$

$(x+1)f(x+1) - (x-1)f(x-1)$

$$= \frac{\quad}{4} - \frac{\quad}{4}$$

$$= \frac{(x+1)\left(f(x)+f(1)\right)}{4} - \frac{(x-1)\left(f(x)-f(1)\right)}{4}$$

$$= \frac{2xf(1)+2f(x)}{4}$$

$2f(x) = xf(1) + f(x)$

$f(x) = xf(1)$

$\therefore \boxed{f(x) = cx \quad \text{for } c \in \mathbb{R}}$