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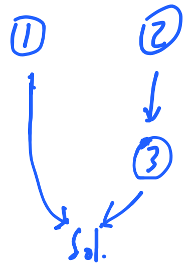
Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$



Let $a = x+y$, $b = xy$.

Rewriting,

$$a^2 - b = \left(\frac{a}{3} + 1\right)^3.$$

$3 \mid a$ because $a^2 - b \in \mathbb{Z}$.

Therefore, let $a = 3k$

$$9k^2 - b = (k+1)^3 = k^3 + 3k^2 + 3k + 1$$

$$\therefore b = -k^3 + 6k^2 - 3k - 1$$

$$t^2 - at + b = 0$$

$\Delta = m^2 \quad (m \in \mathbb{Z})$

$$a^2 - 4b = m^2$$

$$9k^2 + 4k^3 - 24k^2 + 12k + 4 = m^2$$

$$4k^3 - 15k^2 + 12k + 4 = m^2$$

$$(k-2)(4k^2 - 7k - 2) = m^2$$

$$(k-2)(k-2)(4k+1) = m^2$$

$$\therefore 4k+1 = (2n+1)^2 = 4n^2 + 4n + 1 \qquad (n \in \mathbb{Z})$$

$$t = \frac{a \pm (k-2)(2n+1)}{2} = \frac{3(n^2+n) \pm (n^2+n-2)(2n+1)}{2}$$

$$= \frac{3n^2 + 3n + 2n^3 + 3n^2 - 3n - 2}{2} \quad \text{or} \quad \frac{3n^2 + 3n - 2n^3 - 3n^2 + 3n + 2}{2}$$

$$= n^3 + 3n^2 - 1 \quad \text{or} \quad -n^3 + 3n + 1$$

$$\therefore \boxed{(x,y) = (n^3 + 3n^2 - 1, -n^3 + 3n + 1) \text{ or } (-n^3 + 3n + 1, n^3 + 3n^2 - 1) \text{ for } n \in \mathbb{Z}}$$