Enoch Yu

If $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ are all polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x),$$

prove that $x - 1$ is a factor of $P(x)$.

$\Longleftrightarrow$

$P(1) = 0$

①

Let w_1, w_2, w_3, w_4, w_5 be the 5th roots of unity where, WLOG, $w_5 = 1$.

$$P(1) + w_1 Q(1) + w_1^2 R(1) = 0$$
$$P(1) + w_2 Q(1) + w_2^2 R(1) = 0$$
$$P(1) + w_3 Q(1) + w_3^2 R(1) = 0$$
$$P(1) + w_4 Q(1) + w_4^2 R(1) = 0$$

Assume $P(1)$, $Q(1)$, $R(1)$ as variable,

$P(1) = Q(1) = R(1) = 0$ is a solution to the triple.

$$4P(1) + (w_1 + \cdots + w_4)(Q(1) + R(1)) = 0$$

$$\therefore 4P(1) = Q(1) + R(1)$$

②

$$Q(1)\left(\tfrac{1}{4} + w_1\right) + R(1)\left(\tfrac{1}{4} + w_1^2\right) = 0$$
$$Q(1)\left(\tfrac{1}{4} + w_2\right) + R(1)\left(\tfrac{1}{4} + w_2^2\right) = 0$$

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