Enoch Yu

Prove that $x^4 + 1$ is irreducible in $\mathbb{Q}[x]$.

A polynomial is irreducible in integers if it cannot be represented as a multiple of nonconstant integer polynomials.

$$\left.\begin{array}{ll} x^2+2x+1 = (x+1)^2 & \checkmark \\ 2x+1 & \times \\ x^2+1 & \end{array}\right\} \text{ in } \mathbb{Z}[x]$$

$K[x]$ is a set of polynomials composed of monomials with coefficients in the set K.

$\mathbb{R}[x] \Rightarrow$ the set of polynomials in real numbers

e.g. $\sqrt{2}x^2 + \frac{2}{3}x + 19$, πx^2

$\mathbb{Q}[x] \Rightarrow$ " rational "

e.g. $\frac{1}{3}x^3 + 2x - 3$

$\mathbb{Z}[x] \Rightarrow$ " integers

e.g. $2x^2 + 5x$

$C[x] \Rightarrow$ " complex "

e.g. ix^2

Prove that $x^4 + 1$ is irreducible in $\mathbb{Q}[x]$.

pf 1)

Rational Root theorem

If $\frac{p}{q}$ $((p,q)=1)$ is a root of $p(x)$, then, $p \mid a_0$, $q \mid a_n$.

$q = \pm 1$, $p = \pm 1$

$\therefore$ rational root $= \pm 1$ if it exists.

$x^4 + 1 > 0$, x^4+1 has no solution over $\mathbb{R}$

$\Longleftrightarrow$

x^4+1 cannot be written as a multiple of linear and cubic polynomial.

$\therefore x^4+1 = (ax^2+bx+c)(dx^2+ex+f)$

$ad = 1$, $cf = 1$

$a = d = \pm 1$, $c = f = \pm 1$

$x^4+1 = (x^2+bx\pm 1)(x^2+ex\pm 1)$

$= x^4 + (b+e)x^3 + (be \pm 2)x^2 \pm (b+e)x + 1$

$\therefore b = -e$

However, $be = \pm 2$ is unobtainable since $b, e \notin \mathbb{Q}$.

Thus, x^4+1 is irreducible in $\mathbb{Q}[x]$. $\square$

PF 2) Eisenstein's criterion.

Let $p(x) \in \mathbb{Z}[x]$ be defined as $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$.

$p(x)$ is irreducible in $\mathbb{Q}[x]$ if there exists a prime p such that

- $p \nmid a_n$
- $p \mid a_i \quad \forall \quad 0 \le i < n$
- $p^2 \nmid a_0$.

Let $p(x) = x^4+1$.

If $p(x+1)$ is irreducible in $\mathbb{Q}[x]$, $p(x)$ is irreducible in $\mathbb{Q}[x]$.

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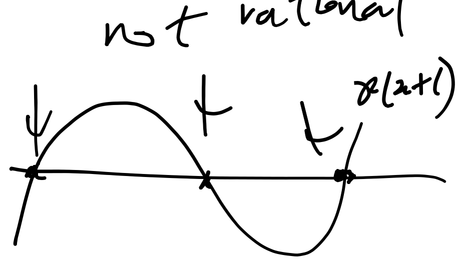

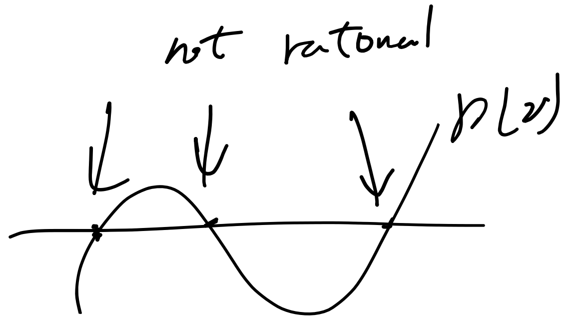

$$p(x+1) = (x+1)^4 + 1 = x^4 + \binom{4}{1}x^3 + \binom{4}{2}x^2 + \binom{4}{3}x + 2$$

Consider $p = 2$.

- $p \mid a_i \ \forall \ 0 \le i < n$
- $p \nmid a_n$
- $p^2 \nmid a_0$

Therefore, by Eisenstein's criterion, x^4+1 is irreducible in $\mathbb{Q}[x]$.

$\square$