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Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$

1

let $S_n = 1 + x_1^2 + \cdots + x_n^2$ where $S_0 = 1$.

$\frac{x_1}{S_1} + \frac{x_2}{S_2} + \cdots + \frac{x_n}{S_n} < \sqrt{n}$ is to be demonstrated.

2

It suffices to prove $\left(\sum_{i=1}^{n} \frac{x_i}{S_i}\right)^2 < n$ where $x_i \geq 0$ because if the inequality holds for $x_i \geq 0$, the original inequality holds for $x_i \in \mathbb{R}$.

$$\left(\sum_{i=1}^{n} \frac{x_i}{S_i}\right)\left(\sum_{i=1}^{n} \frac{x_i}{S_i}\right)$$

Since addition is commutative, we can rearrange the orders of $\frac{x_i}{S_i}$ such that $\frac{x_a}{S_a} \geq \frac{x_b}{S_b} \geq \cdots \geq \frac{x_z}{S_z}$ where $\{a, b, \ldots, z\} = \{1, 2, \ldots, n\}$.

3

After rearrangement,

$$n\left(\sum_{i=1}^{n} \frac{x_i^2}{S_i^2}\right) \geq \left(\sum_{i=1}^{n} \frac{x_i}{S_i}\right)\left(\sum_{i=1}^{n} \frac{x_i}{S_i}\right)$$

holds by Chebyshev's inequality.

Therefore, it suffices to prove that

$$\sum_{i=1}^{n} \frac{x_i^2}{S_i^2} < 1$$

Since if $\sum^{n} \frac{x_i^2}{S_i^2} < 1$, then, $\left(\sum_{i=1}^{n} \frac{x_i}{S_i}\right)^2 \leq n\left(\sum_{i=1}^{n} \frac{x_i^2}{S_i^2}\right) < n$.

Notice that $S_n - S_{n-1} = x_n^2$ and $S_i^2 \geq S_i S_{i-1}$. $\leftarrow$ (‡)

Continuing,

$$\sum_{i=1}^{n} \frac{x_i^2}{S_i^2} \leq \sum_{i=1}^{n} \frac{S_i - S_{i-1}}{S_i S_{i-1}} = \sum_{i=1}^{n} \left(\frac{1}{S_{i-1}} - \frac{1}{S_i} \right) = \frac{1}{S_0} - \frac{1}{S_n} < 1.$$

$\square$