Enoch Yu

Find all polynomials f with real coefficients such that for all reals a, b, c such that $ab + bc + ca = 0$ we have the following relations

$$f(a-b) + f(b-c) + f(c-a) = 2f(a+b+c).$$

$a=b=c=0$

$f(0)+f(0)+f(0)=2f(0)$

$\therefore f(0)=0$

$b=c=0$

$f(a)+f(0)+f(-a)=2f(a)$

$f(a)=f(-a)$

$\therefore f(x)$ is even.

$a=\frac{-bc}{b+c}$

$(a,b,c)=(2x,2x,-x)$ X

$(a,b,c)=(6x,3x,-2x)$

$f(3x)+f(5x)+f(8x)=2f(7x)$

let $f(x)=\alpha x^2+\beta x^4+\cdots+\omega x^n$

$\alpha(3x)^2+\beta(3x)^4+\cdots+\omega(3x)^n$
$+\alpha(5x)^2+\beta(5x)^4+\cdots+\omega(5x)^n$
$+\alpha(8x)^2+\beta(8x)^4+\cdots+\omega(8x)^n$

$\alpha\left((3x)^2+(5x)^2+(8x)^2\right)+\cdots+\omega\left((3x)^n+(5x)^n+(8x)^n\right)$

$=\alpha\left(2(7x)^2\right)+\cdots+\omega\left(2(7x)^n\right)$

$(3x)^k+(5x)^k+(8x)^k=2(7x)^k$

$3^k+5^k+8^k=2\cdot7^k$

$\left(\frac{3}{7}\right)^k+\left(\frac{5}{7}\right)^k+\left(\frac{8}{7}\right)^k=2$

$\left(\frac{8}{7}\right)^k \geq 2$

$8^5=2^{15}=1024\cdot32$
$=32768$

1024
32

2048
30720

32768

$7^5=343\cdot49$
$=16150-343$
$=16807$

$16807\cdot2=33614$

$\cancel{343}$
$\cancel{2401}$

512
512

1024
512
2560

$8^6=2^{18}=512^2=262144$

$7^6=117549$

16807
7

$k<6$

$k=2,4$

4921

i) $k=2$

$9+25+64=2\cdot49$
$98=98$

ii) $k=4$ 4802

$81+625+4096=2\cdot2401$
$=4802$