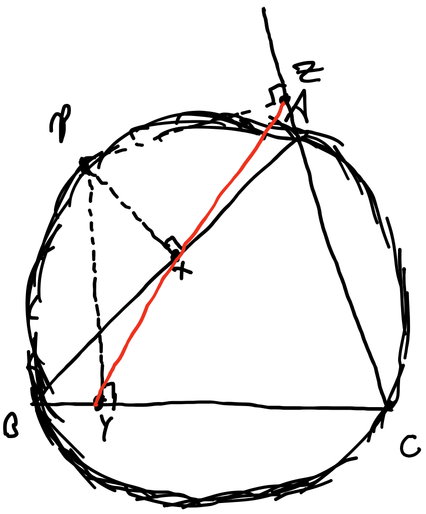
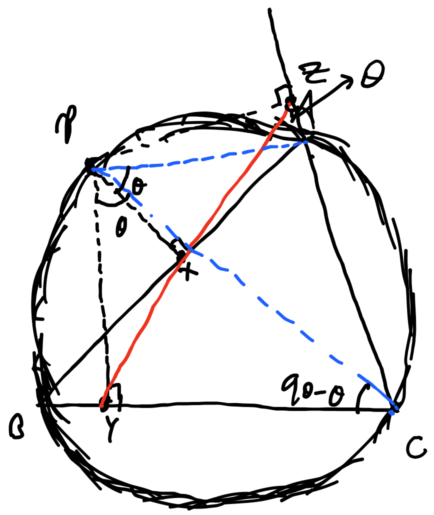
Enoch Yu

(Simson Line) Let there be a point P and $\triangle ABC$ on the same plane. Define points X, Y, Z as the intersections of the foots from P to $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ respectively. Point P is on the circumcircle of $\triangle ABC$ if and only if X, Y, Z are collinear. Then, the line that passes through X, Y, Z are called Simson line.



Proof of the existance of Simson line.

i) If point P is on the circumcircle of $\triangle ABC$, then, X, Y, Z are collinear.



Let $\angle YPC = \theta$.

$\angle PCY = 90 - \theta = \angle PCB = \angle PAB = \angle PAX$

Notice that $\angle PZA = \angle PXA = 90°$

$\therefore \square PZAX$ is cyclic.

Therefore,

$\angle XZA = \theta$

Moreover, $\angle PYC = \angle PZC = 90°$

$\therefore \square PYCZ$ is cyclic.

Thus, $\angle YPC = \angle YZC = \theta$

Because $\angle XZA = \angle YZC = \theta$, X, Y, Z are collinear.

ii) If X, Y, Z are collinear, then point P is on the cir

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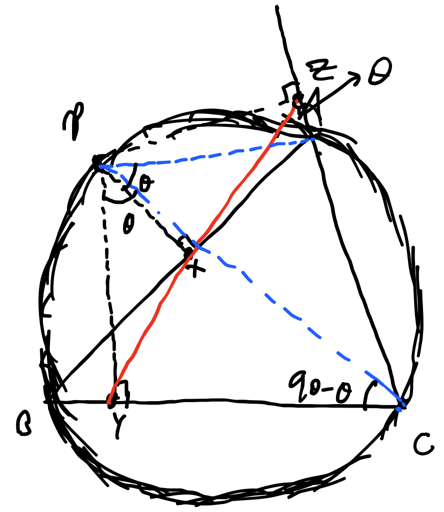
1.) If X, Y, Z are collinear, then point P lies on the circumcircle of $\triangle ABC$.



Let $\angle YZC = \angle XZA = \theta$.

Because $\angle PYC = \angle PZC = 90°$, $\square PYCZ$ is cyclic.

Thus, $\angle YPC = \theta$ and $\angle PCY = 90-\theta$.

Similarly, because $\angle PXA = \angle PZA = 90°$, $\square PXAZ$ is cyclic.

Thus, $\angle XPA = \theta$ and $\angle PAX = 90-\theta$.

$\angle PCY = \angle PCB = 90-\theta = \angle PAX = \angle PAB$

$\therefore \angle PCB = \angle PAB$.

Therefore point P lies on the circumcircle of $\triangle ABC$.

$\square$