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Let $a_2, \ldots, a_n$ be $n-1$ positive real numbers, where $n \ge 3$, such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3\cdots(1+a_n)^n > n^n.$$

$b_i > 0$

1

$a_2 = \frac{b_2}{b_1}$

$a_3 = \frac{b_3}{b_2}$

$\vdots$

$a_{n-1} = \frac{b_{n-1}}{b_{n-2}}$

$a_n = \frac{b_1}{b_{n-1}}$

$$\left(1+\frac{b_2}{b_1}\right)^2 \cdot \left(1+\frac{b_3}{b_2}\right)^3 \cdots \left(1+\frac{b_1}{b_{n-1}}\right)^n > n^n$$

$$(b_1+b_2)^2(b_2+b_3)^3 \cdots (b_{n-1}+b_n)^n > n^n b_1^2 b_2^3 \cdots b_{n-1}^n$$

$$(b_1+b_2)^2 \ge 2^2 b_1 b_2$$

$$\left(\frac{b_2}{2}+\frac{b_2}{2}+b_3\right)^3 \ge 3^3 \cdot \frac{b_2}{2}\cdot\frac{b_2}{2}\cdot b_3$$

2

$$\vdots$$

$$\left(\frac{b_{n-1}}{n-1}+\cdots+\frac{b_{n-1}}{n-1}+b_n\right)^n \ge n^n \cdot \left(\frac{b_{n-1}}{n-1}\right)^{n-1} b_1$$

$$(b_1+b_2)^2(b_2+b_3)^3\cdots(b_{n-1}+b_n)^n \ge \cancel{2}^2 b_1 b_2 \cdot \cancel{3}^3 \cdot \frac{b_2}{\cancel{2}}\cdot\frac{b_2}{\cancel{2}}\cdot b_3 \cdots n^n \left(\frac{b_{n-1}}{\cancel{n-1}}\right)^{n-1} b_1$$

$$= n^n b_1^2 b_2^3 b_3^4 \cdots b_{n-1}^n$$

$$b_1 = b_2, \quad \frac{b_2}{2} = b_3, \quad \ldots, \quad \frac{b_{n-1}}{n-1} = b_1$$

$$b_1 = b_2, \quad b_2 = 2b_3, \quad b_3 = 3b_4, \quad \ldots, \quad b_{n-1} = (n-1)b_1$$

$$b_1 = b_2 = 2b_3 = 2\cdot 3 b_4 = 2\cdot 3\cdot 4 b_5 = \cdots = 2\cdot 3\cdot 4 \cdots (n-1) b_1$$

$$b_1 = (n-1)!\, b_1$$

3

$$(n-1)! = 1$$

$$\therefore n = 1 \text{ or } n = 2.$$

Because $n \ge 3$, the equality does not satisfy.

The steps are reversible.

$\square$