Enoch Yu

Find all numbers $n \geq 3$ for which there exists real numbers $a_1, a_2, ..., a_{n+2}$ satisfying $a_{n+1} = a_1, a_{n+2} = a_2$ and

$$a_i a_{i+1} + 1 = a_{i+2}$$

for $i = 1, 2, ..., n$.

$$a_i a_{i+1} a_{i+2} + a_{i+2} = a_{i+2}^2 \qquad (i \geq 1)$$

$$a_{i-1} a_i a_{i+1} + a_{i-1} = a_{i-1} a_{i+2} \qquad (i \geq 2)$$

$$a_i a_{i+1} a_{i+2} + a_i = a_i a_{i+3} \qquad (i \geq 1)$$

1

$$\sum_{i=1}^{n} (a_i a_{i+1} a_{i+2} + a_{i+2}) = \sum_{i=1}^{n} a_{i+2}^2$$

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$$\sum_{i=1}^{n} a_{i+2} = \sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} (a_i a_{i+1} a_{i+2} + a_i) = \sum_{i=1}^{n} a_i a_{i+3}$$

$$\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} a_{i+3}^2$$

$$\therefore \sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} a_i a_{i+3}$$

$$\sum_{i=1}^{n} (a_i^2 - a_i a_{i+3}) = 0$$

Assuming that $a_{n+3} = a_3$,

$$\frac{1}{2} \sum_{i=1}^{n} (a_i^2 - 2a_i a_{i+3} + a_{i+3}^2) = 0$$

$$\sum_{i=1}^{n} (a_i - a_{i+3})^2 = 0$$

$$\therefore a_i = a_{i+3}$$

3

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	$\cdots$
-1	-1	2	-1	-1	2	-1	-1	2	

If $n = 3k$,

$$(a_i, a_{i+1}, a_{i+2}) = (-1, -1, 2)$$

$$\boxed{n \text{ is a multiple of } 3.}$$

If $n \neq 3k$

i) $n = 3k+1$ $\quad a_{3k+2} = a_1 \quad a_{3k+3} = a_2$

$$a_{3k+3} = a_3 = a_2 = a_{3k+2} = a_1$$

ii)

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ii) $n = 3k+2$ $\quad a_{3k+3} = a_1$, $\quad a_{3k+4} = a_2$

$$a_3 = a_{3k+3} = a_1 = a_{3k+1} = a_{3k+4} = a_2$$

$$\therefore\ a_1 = a_2 = \cdots = a_{n+2} = a$$

$$a^2 + 1 = a$$

$$a^2 - a + 1 = 0$$

$$a = \frac{1 \pm \sqrt{-3}}{2} \notin \mathbb{R}$$