Enoch Yu

Determine the value of

$$S=\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\cdots+\sqrt{1+\frac{1}{1999^2}+\frac{1}{2000^2}}$$

$$1+\frac{1}{n^2}+\frac{1}{(n+1)^2}=(\qquad)^2 \qquad \frac{1}{n}-\frac{1}{n+1}=\frac{1}{n(n+1)} \qquad 1$$

$$\left(1+\frac{1}{n}+\frac{1}{n+1}\right)^2=1+\frac{1}{n^2}+\frac{1}{(n+1)^2}+2\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n(n+1)}\right)$$

$$\left(1+\frac{1}{n}-\frac{1}{n+1}\right)^2=1+\frac{1}{n^2}+\frac{1}{(n+1)^2}+2\left(\cancel{\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n(n+1)}}\right) \qquad 2$$

$$S=\sum_{i=1}^{1999}\left(1+\frac{1}{i}-\frac{1}{i+1}\right)$$

$$=1999+\sum_{i=1}^{1999}\left(\frac{1}{i}-\frac{1}{i+1}\right)$$

$$=1999+1-\frac{1}{2000}$$

$$=2000-\frac{1}{2000}$$

$$=\frac{2000^2-1}{2000}$$

$$=\boxed{\frac{3999999}{2000}}$$