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Let $n \geq 2$ be an integer, and let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{1-a_k}(a_1 + a_2 + \cdots + a_{k-1})^2 < \frac{1}{3}.$$

$S_k = a_1 + a_2 + \cdots + a_k \quad (S_0 = 0), \quad S_n = 1$

$$\sum_{k=1}^{n}(S_k - S_{k-1}) = 1$$

$$\sum_{k=1}^{n}(S_k^2 - S_{k-1}^2) = 1$$

$$\sum_{k=1}^{n}(S_k^3 - S_{k-1}^3) = 1$$

$$\sum_{k=1}^{n} \frac{a_k S_{k-1}^2}{1-a_k} < \frac{1}{3}$$

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$$\sum_{k=1}^{n} \frac{a_k S_{k-1}^2}{1-a_k} < \frac{1}{3}\sum_{k=1}^{n}(S_k^3 - S_{k-1}^3)$$

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let $\alpha = a_k, \quad \beta = S_{k-1}$

$\alpha + \beta = S_k \leq 1 \qquad \therefore \alpha + \beta \leq 1$

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If $\dfrac{a_k S_{k-1}^2}{1-a_k} < \dfrac{S_k^3 - S_{k-1}^3}{3}$, then $\displaystyle\sum_{k=1}^{n} \frac{a_k S_{k-1}^2}{1-a_k} < \frac{1}{3}\sum_{k=1}^{n}(S_k^3 - S_{k-1}^3)$.

$$\frac{\alpha\beta^2}{1-\alpha} < \frac{(\alpha+\beta)^3 - \beta^3}{3} = \frac{\alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2}{3}$$

$$\frac{\alpha\beta^2}{1-\alpha} - \alpha\beta^2 < \frac{\alpha^3 + 3\alpha^2\beta}{3}$$

$$\frac{\alpha^2\beta^2}{1-\alpha} < \frac{\alpha^3 + 3\alpha^2\beta}{3}$$

$$\frac{\beta^2}{1-\alpha} < \frac{\alpha + 3\beta}{3}$$

$\beta \leq 1 - \alpha \qquad \dfrac{\beta^2}{1-\alpha} \leq \beta$

$$\frac{\beta^2}{1-\alpha} \leq \beta < \frac{\alpha}{3} + \beta$$

$\square$

Let $n \geq 2$ be an integer, and let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{1-a_k}(a_1 + a_2 + \cdots + a_{k-1})^2 < \frac{1}{3}.$$

let $S_k = a_1 + a_2 + \cdots + a_k$

$$\sum^{n} \frac{a_k S_{k-1}^2}{\ldots} < \frac{1}{3}$$

$k=1 \quad 1-a_k \qquad)$

$$\sum_{k=1}^{n} \frac{a_k S_{k-1}^2}{1-a_k} < \frac{1}{3} \sum_{k=1}^{n} (S_k^2 - S_{k-1}^2)$$

If $\frac{a_k S_{k-1}^2}{1-a_k} < \frac{S_k^2 - S_{k-1}^2}{3}$, then we prove our inequality.

let $\alpha = a_k$, $\beta = S_{k-1}$

$$\frac{\boxed{\alpha \beta^2}}{1-\alpha} < \frac{(\alpha+\beta)^2 - \beta^2}{3} = \frac{\alpha^2 + 2\alpha\beta}{3}$$