Enoch Yu

Prove that if a, b, and c are positive real numbers, then

$$a^a b^b c^c \geq (abc)^{(a+b+c)/3}$$

Rearrangement Inequality

If $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq a_{\pi(1)}b_{\pi(1)} + \cdots + a_{\pi(n)}b_{\pi(n)} \geq a_1b_n + \cdots + a_nb_1$$

when $\pi(i)$ is the i^{th} permutation of 1 to n.

Chebyshev's Inequality

$$n\left(\sum_{i=1}^{n} a_ib_i\right) \geq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right)$$

$$a_1b_1 + \cdots + a_nb_n \geq a_1b_1 + \cdots + a_nb_n$$
$$a_1b_1 + \cdots + a_nb_n \geq a_1b_2 + \cdots + a_nb_1$$
$$\vdots$$
$$+)\ a_1b_1 + \cdots + a_nb_n \geq a_1b_n + \cdots + a_nb_1$$
$$n(a_1b_1 + \cdots + a_nb_n) \geq (a_1 + \cdots + a_n)(b_1 + \cdots + b_n)$$

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}} \quad \text{where } a, b, c \in \mathbb{R}^+$$

$$\lg a^a b^b c^c \geq \lg (abc)^{\frac{a+b+c}{3}}$$ 1

$$a\lg a + b\lg b + c\lg c \geq \frac{a+b+c}{3}(\lg a + \lg b + \lg c)$$ 2

$$3(a\lg a + b\lg b + c\lg c) \geq (a+b+c)(\lg a + \lg b + \lg c)$$

WLOG, let $a \geq b \geq c$. Then $\lg a \geq \lg b \geq \lg c$.

By Chebyshev's Inequality, the inequality holds.

$\square$