Enoch Yu

Find all triples (x, y, z) where x, y, z are distinct positive integer that satisfy the following equation.

$$\frac{1}{x+1}+\frac{1}{y+2}+\frac{1}{z+3}=\frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right)$$

let $\overset{c}{a}=x+1$, $\overset{b}{b}=y+2$, $\overset{a}{c}=z+3$ where $a, b, c \geq 2$

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{11}{12}$ ← 1

WLOG, let $a \geq b \geq c$ ← 2

$\frac{3}{a}=\frac{1}{a}+\frac{1}{a}+\frac{1}{a} \leq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{11}{12} \leq \frac{1}{c}+\frac{1}{c}+\frac{1}{c}=\frac{3}{c}$ $\qquad \frac{11}{12}>\frac{3}{4}$

i) $c=2$

$\frac{1}{a}+\frac{1}{b}=\frac{5}{12}$

$12(a+b)=5ab$

$25ab-60(a+b)=0$

$(5a-12)(5b-12)=144$

144	1	×
72	2	×
48	3	(12, 3)
36	4	×
24	6	×
18	8	(6, 4)
16	9	×
12	12	×

ii) $c=3$

$\frac{1}{a}+\frac{1}{b}=\frac{7}{12}$

$12(a+b)=7ab$

$49ab-84(a+b)=0$

$(7a-12)(7b-12)=144$

144	1	×
72	2	(12, 2) ×
48	3	×
36	4	×
24	6	×
18	8	×
16	9	(4, 3)
12	12	×

$(1, 2, 3)$

(a, b, c)	(x, y, z)
• ~~(12, 3, 2)~~	~~(11, 1, −1)~~
~~(12, 2, 3)~~	~~(11, 0, 0)~~
~~(3, 2, 12)~~	~~(2, 0, 9)~~
~~(3, 12, 2)~~	~~(2, 10, −1)~~
~~(2, 3, 12)~~	~~(1, 1, 9)~~
~~(2, 12, 3)~~	~~(1, 10, 0)~~
• ~~(6, 4, 2)~~	~~(5, 2, −1)~~
~~(6, 2, 4)~~	~~(5, 0, 1)~~
~~(4, 6, 2)~~	~~(3, 4, −1)~~
~~(4, 2, 6)~~	~~(3, 0, 3)~~
(2, 4, 6)	(1, 2, 3)
~~(2, 6, 4)~~	~~(1, 4, 1)~~
• ~~(4, 3, 3)~~	
~~(3, 4, 3)~~	
~~(3, 3, 4)~~	