Enoch Yu

Show that, for any fixed integer $n \geq 1$, the sequence

$$2,\ 2^2,\ 2^{2^2},\ 2^{2^{2^2}}, \ldots \pmod n$$

is eventually constant.

[The tower of exponents is defined by $a_1 = 2$, $a_{i+1} = 2^{a_i}$. Also $a_i \pmod n$ means the remainder which results from dividing a_i by n.]

① ↙

FTSOC, let "a" be the minimum value of "n" such that
$2, 2^2, 2^{2^2}, \ldots \pmod a$ is not constant.

Since $2, 2^2, 2^{2^2}, \ldots \pmod a$ is not constant, let "k" be the period of the sequence.

$\therefore\ 1 \leq k \leq a-1 < a$ holds.

② ↙

Because $2, 2^2, 2^{2^2}, \ldots \pmod a$ is periodic, $1, 2, 2^2, 2^{2^2}, \ldots \pmod k$ must also be periodic.

Therefore, $1, 2, 2^2, 2^{2^2}, \ldots \pmod k$ is not constant,

As a result, $2, 2^2, 2^{2^2}, \ldots \pmod k$ is also not constant.

However, $k < a$ and "a" was assumed to be the least value of "k" such that $2, 2^2, 2^{2^2}, \ldots \pmod a$ is not constant. Thus, contradiction is present.

$\square$