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The real numbers a, b, c, d are such that $a \ge b \ge c \ge d > 0$ and $a+b+c+d=1$. Prove that

$$(a+2b+3c+4d)a^a b^b c^c d^d < 1.$$

Variable	a	b	c	d
Weight	a	b	c	d

1

$$\frac{a\cdot a+b\cdot b+c\cdot c+d\cdot d}{a+b+c+d} \ge \sqrt[a+b+c+d]{a^a b^b c^c d^d}$$

$$a^2+b^2+c^2+d^2 \ge a^a b^b c^c d^d$$

$$(a+2b+3c+4d)a^a b^b c^c d^d \le \underbrace{(a+2b+3c+4d)}_{X}(a^2+b^2+c^2+d^2)$$

$$= a^2X+b^2X+c^2X+d^2X$$

$$X \le a+3b+3c+3d$$

2

$$\cancel{a}+2\cancel{b}+\cancel{3c}+4d \le \cancel{a}+\cancel{3b}+\cancel{3c}+3d$$

$$0 \le b-d$$

$$X \le 3a+b+3c+3d$$

$$X \le 3a+3b+c+3d$$

$$X \le 3a+3b+3c+d$$

$$X(a^2+b^2+c^2+d^2) \le a^2(a+3b+3c+3d)+b^2(3a+b+3c+3d)+c^2(3a+3b+c+3d)+d^2(3a+3b+3c+d)$$

$$= a^3+b^3+c^3+d^3+3a^2(b+c+d)+3b^2(c+d+a)+3c^2(d+a+b)+3d^2(a+b+c)$$

$$< (a+b+c+d)^3 = 1$$

$$(a+b+c+d)^3 = a^3+b^3+c^3+d^3+3a^2(b+c+d)+3b^2(c+d+a)+3c^2(d+a+b)+3d^2(a+b+c)+6abc+6abd+6acd+6bcd$$

$\therefore\ (a+2b+3c+4d)a^a b^b c^c d^d < 1$

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