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Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all real numbers x, y.

i) let $x = y = 0$ and $f(0) = k$

$f(-k) = f(k) + k - 1$

$\cancel{f(k)} = \cancel{f(k)} + k - 1$

$\therefore k = 1$

ii) let $x = f(y)$

$f(0) = f(x) + x^2 + f(x) - 1$

$f(x) = -\dfrac{x^2 - 1 - k}{2}$

1

$f(x)$ is even ($\because f(k) = f(-k)$)

$\therefore f(k) = f(-k)$

$$\boxed{f(x) = \frac{2 - x^2}{2}}$$

$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$

LHS: $f\left(x - \dfrac{2-y^2}{2}\right) = \dfrac{2 - \left(x - \frac{2-y^2}{2}\right)^2}{2} = \cancel{1} - \dfrac{\cancel{x^2} - \cancel{x(2-y^2)} + \cancel{\frac{y^4 - 4y^2 + 4}{4}}}{2}$

$\parallel$

RHS: $\dfrac{2 - \left(\frac{2-y^2}{2}\right)^2}{2} + x \cdot \dfrac{2-y^2}{2} + \dfrac{2-x^2}{2} - 1 = \cancel{1} - \dfrac{\cancel{\frac{y^4 - 4y^2 + 4}{4}}}{2} + \cancel{\dfrac{2x - xy^2}{2}} \cancel{+1} - \cancel{\dfrac{x^2}{2}} \cancel{-1}$