Enoch Yu

For $n \geq 2$ let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \leq \left(n + \frac{1}{2}\right)^2.$$

Prove that $\max(a_1, a_2, \ldots, a_n) \leq 4\min(a_1, a_2, \ldots, a_n)$.

WLOG, let $a_1 \geq a_2 \geq \cdots \geq a_n$

$$(a_n + \underbrace{a_2 + \cdots + a_{n-1}}_{n-2} + a_1)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) \geq \left(\sqrt{\frac{a_n}{a_1}} + n-2 + \sqrt{\frac{a_1}{a_n}}\right)^2$$

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$$\left(n+\frac{1}{2}\right)^2 \geq \left(\sqrt{\frac{a_n}{a_1}} + \sqrt{\frac{a_1}{a_n}} + n-2\right)^2$$

$$\cancel{n}+\frac{1}{2} \geq \cancel{n}-2+\sqrt{\frac{a_n}{a_1}} + \sqrt{\frac{a_1}{a_n}}$$

$$\frac{25}{4} \geq \frac{a_n}{a_1} + \frac{a_1}{a_n} + 2$$

$$\frac{17}{4} \geq \frac{a_n}{a_1} + \frac{a_1}{a_n}$$

$$\frac{17}{4}a_1a_n \geq a_n^2 + a_1^2 \qquad 4a_1 - a_n > 0$$

$$4a_1^2 + 4a_n^2 - 17a_1a_2 \leq 0$$

$$(4a_1 - a_n)(a_1 - 4a_n) \leq 0$$

$$a_1 \leq 4a_n$$

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