Enoch Yu

Suppose that $a$, $b$, $c$, $d$ are positive real numbers satisfying $(a+c)(b+d) = ac+bd$. Find the smallest possible value of

$$S = \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

$a=b=c=d$

$2a \cdot 2a = a^2+a^2$

$4a^2 = 2a^2$

$\therefore a=0 \Rightarrow X$

$a=b=c,\ d$

$2a(a+d) = a^2+ad$

$a^2+ad=0$

$a=0$ or $a=-d$

$1+1+\frac{a}{d}+\frac{d}{a}$

$a=b$ and $c=d$

or

$a=c$ and $b=d$

$(a+c)^2 = ac+ac$

$a^2+c^2=0$ X $a=c=0$

$2a \cdot 2b = a^2+b^2$

$4ab = a^2+b^2$

$\frac{4b}{a} = 1+\frac{b^2}{a^2}$

let $x=\frac{b}{a}$

$x^2-4x+1=0$

$x=2\pm\sqrt{3}$

$\frac{a}{b}+\frac{b}{a}+\frac{a}{b}+\frac{b}{a} = 2\left(\frac{a}{b}+\frac{b}{a}\right) = 8$

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$2+\sqrt{3}+\frac{1}{2+\sqrt{3}} = 2-\sqrt{3}+\frac{1}{2-\sqrt{3}}$

$2\cancel{+\sqrt{3}}+2\cancel{-\sqrt{3}} = 2\cancel{-\sqrt{3}}+2\cancel{+\sqrt{3}}$

$4=4$

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1 2 3

$$S = \frac{a}{b}+\frac{c}{d}+\frac{b}{c}+\frac{d}{a} \geq 2\sqrt{\frac{ac}{bd}}+2\sqrt{\frac{bd}{ca}} = 2\cdot\frac{ac+bd}{\sqrt{abcd}} \geq \frac{2(a+c)(b+d)}{\sqrt{abcd}}$$

$$\geq \frac{2\cdot 2\cancel{\sqrt{ac}}\cdot 2\cancel{\sqrt{bc}}}{\cancel{\sqrt{abcd}}}$$

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