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Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\le 8.$$

$$\sum_{cyc}\frac{(2a+b+c)^2}{2a^2+(b+c)^2}=\sum_{cyc}\frac{4a^2+b^2+c^2+4ab+2bc+4ca}{2a^2+b^2+c^2+2bc}$$

$$=\sum_{cyc}\left(1+\frac{2a^2+4a(b+c)}{2a^2+(b+c)^2}\right)=\sum_{cyc}\frac{2a^2+4a(b+c)}{2a^2+(b+c)^2}+3$$ 1

$$=\sum_{cyc}\left(1+\frac{(b+c)(4a-b-c)}{2a^2+(b+c)^2}\right)+3=\sum_{cyc}\frac{(b+c)(4a-b-c)}{2a^2+(b+c)^2}+6$$

$$\frac{2a\cdot\cancel{2a}}{2a^2+\cancel{4a^2}}\cdot 3=2$$

$\square$

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let $a+b+c=k$

$$\sum_{cyc}\frac{(a+k)^2}{2a^2+(k-a)^2}=\sum_{cyc}\frac{a^2+k^2+2ak}{3a^2-2ak+k^2}=\sum_{cyc}\frac{\frac13(3a^2-2ak+k^2)+\frac{8ak}{3}+\frac{2k^2}{3}}{3a^2-2ak+k^2}$$

$$=\sum_{cyc}\left(\frac13+\frac{\frac{8ak}{3}+\frac{2k^2}{3}}{3a^2-2ak+k^2}\right)=\sum_{cyc}\frac{8ak+2k^2}{9a^2-6ak+3k^2}+1$$

$$\frac{8ak+2k^2}{9a^2-6ak+3k^2}=\frac{8ak+2k^2}{(3a-k)^2+2k^2}\le\frac{8ak+2k^2}{2k^2}=\frac{4a+k}{k}$$

1

$$\sum_{cyc}\frac{8ak+2k^2}{9a^2-6ak+3k^2}+1\le\sum_{cyc}\frac{4a+k}{k}+1=\sum_{cyc}\left(1+\frac{4a}{k}\right)+1$$

$$=\sum_{cyc}\frac{4a}{k}+4$$

$$= \frac{4(a+b+c)}{k} + 4$$

$$= 8$$

$\square$