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Let a, b, c be positive real numbers. Prove that $\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1$.

$$\sum_{cyc} \frac{a}{\sqrt{a^2+8bc}} \geq 1$$

$$(a_1+\cdots+a_n)^{\lambda_a}(b_1+\cdots+b_n)^{\lambda_b}\cdots(z_1+\cdots z_n)^{\lambda_z} \geq \sum_{i=1}^{n} a_i^{\lambda_a} b_i^{\lambda_b} \cdots z_i^{\lambda_z}$$

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$$\left(\sum_{cyc} a(a^2+8bc)\right)^{\frac{1}{3}}\left(\sum_{cyc} \frac{a}{\sqrt{a^2+8bc}}\right)^{\frac{1}{3}}\left(\sum_{cyc} \frac{a}{\sqrt{a^2+8bc}}\right)^{\frac{1}{3}} \geq a+b+c$$

$$\left(\sum_{cyc} a(a^2+8bc)\right)\left(\sum_{cyc} \frac{a}{\sqrt{a^2+8bc}}\right)^{2} \geq (a+b+c)^3 \geq \sum_{cyc} a(a^2+8bc)$$

If $\left(\sum_{cyc} a(a^2+8bc)\right)\left(\sum_{cyc} \frac{a}{\sqrt{a^2+8bc}}\right)^{2} \geq \sum_{cyc} a(a^2+8bc)$, then

$$\sum_{cyc} \frac{a}{\sqrt{a^2+8bc}} \geq 1.$$

a^3, a^2b, abc

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$$(a+b+c)^3 \geq \sum_{cyc} a(a^2+8bc)$$

$$\cancel{a^3}+\cancel{b^3}+\cancel{c^3}+3a^2(b+c)+3b^2(c+a)+3c^2(a+b)+6abc \geq \cancel{a^3}+8abc+\cancel{b^3}+8abc+\cancel{c^3}+8abc$$

$$3a^2(b+c)+3b^2(c+a)+3c^2(a+b)-18abc \geq 0$$

$$a^2(b+c)+b^2(c+a)+c^2(a+b)-6abc \geq 0$$

$$a^2(b+c)-2abc+b^2(c+a)-2abc+c^2(a+b)-2abc \geq 0$$

$$a(b^2+c^2-2bc)+b(a^2+c^2-2ac)+c(a^2+b^2-2ab) \geq 0$$

$$a(b-c)^2+b(a-c)^2+c(a-b)^2 \geq 0 \Rightarrow \text{True}$$

$\square$