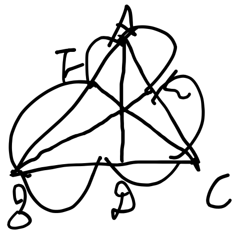
Enoch Yu

Ceva's Theorem

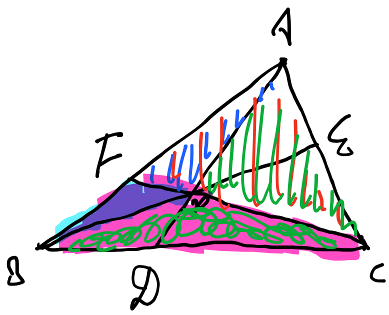
Let P be a point inside a triangle ABC. Define the points D, E, F as following: $D := BC \cap AP$, $E := CA \cap BP$, $F := AB \cap CP$. If AD, BE, and CF concur at point P, then,

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

The converse is also true, $\Leftrightarrow$ If $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$, then AD, BE, and CF concur.



pf)



WLOG, find $\frac{AF}{FB}$.

$$\frac{a}{b} = \frac{c}{d} = \frac{a \pm c}{b \pm d}$$

$$\frac{AF}{FB} = \frac{[APF]}{[FPB]} = \frac{[AFC]}{[FCB]} = \frac{[AFC] - [APF]}{[FCB] - [FPB]} = \frac{[APC]}{[PCB]}$$

Similarly

$$\frac{BD}{DC} = \frac{[APB]}{[APC]} \quad \text{and} \quad \frac{CE}{EA} = \frac{[BPC]}{[APB]}$$

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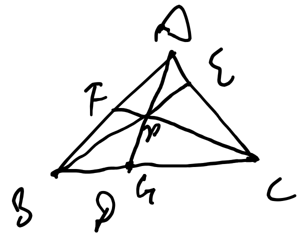
$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{\cancel{[APC]}}{\cancel{[PCB]}}\cdot\frac{\cancel{[APB]}}{\cancel{[APC]}}\cdot\frac{\cancel{[BPC]}}{\cancel{[APB]}}=1$$



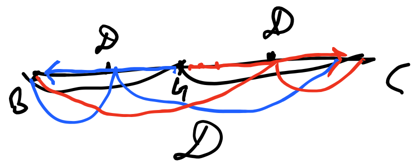
$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1$. We want to prove that AD, BC, CF concur

let $P = BE \cap CF$ and $G = AP \cap BC$.

By our previous statement,

$$\cancel{\frac{AF}{FB}}\cdot\frac{BG}{GC}\cdot\cancel{\frac{CE}{EA}}=\cancel{\frac{AF}{FB}}\cdot\frac{BD}{DC}\cdot\cancel{\frac{CE}{EA}}$$

$$\frac{BG}{GC}=\frac{BD}{DC}$$



$\therefore G = D$

By construction, AD, BE, CF concur.

$\blacksquare$