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Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

1

$a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$

$abc = \frac{1}{xyz} = 1$

$\therefore xyz = 1$

$$\frac{x^3}{\frac{y+z}{yz}} + \frac{y^3}{\frac{z+x}{zx}} + \frac{z^3}{\frac{x+y}{xy}}$$

$$= \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}$$

Titu's lemma

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1+a_2+\cdots+a_n)^2}{b_1+b_2+\cdots+b_n}$$

2

$$(x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n)^2 \geq (x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2$$

let $x_i = \sqrt{b_i}$, $y_i = \frac{a_i}{\sqrt{b_i}}$

$$(b_1 + b_2 + \cdots + b_n)\left(\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n}\right) \geq (a_1 + a_2 + \cdots + a_n)^2$$

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lemma

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}$$

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{(x+y+z)^2}{2(x+y+z)} = \frac{x+y+z}{2}$$

$$x+y+z \geq 3\sqrt[3]{xyz} = 3$$

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{x+y+z}{2} \geq \frac{3}{2}$$

$\square$