Enoch Yu

Compute the sum $S = \sum_{i=0}^{101} \frac{x_i^3}{1-3x_i+3x_i^2}$ for $x_i = \frac{i}{101}$.

$$\frac{x_i^3}{(1-x_i)^3 + x_i^3} = \frac{\left(\frac{i}{101}\right)^3}{\left(1-\frac{i}{101}\right)^3 + \left(\frac{i}{101}\right)^3} = \frac{i^3}{(101-i)^3+i^3}$$

$$= 1 - \frac{(101-i)^3}{(101-i)^3+i^3}$$

0	101
1	100
⋮	
50	51

①

$$2S = \sum_{i=0}^{101}\left(1 + \frac{i^3-(101-i)^3}{(101-i)^3+i^3}\right)$$

$$= 102$$

$\therefore \ S = 51$

let $a, b \in \mathbb{N}_0$ s.t. $a+b=101$.

then,

$$\frac{a^3-(101-a)^3}{(101-a)^3+a^3} + \frac{b^3-(101-b)^3}{(101-b)^3+b^3} = 0$$

$$\frac{\cancel{a^3}-(\cancel{101-a})^3}{(101-a)^3+a^3} + \frac{(101-\cancel{a})^3-\cancel{a^3}}{a^3+(101-a)^3} = 0$$