Enoch Yu

There are 999 mysterious Thinking Trees, each labeled uniquely from 1 to 999. According to the legends, if a boy waters the tree(s) randomly, and if the sum of the number of watered tree(s) is divisible by 3, a magical seed will fall from the sky. A young boy Enoch heard about the legend, and wanted to make his own magical farm. How many seeds can Enoch receive by watering the trees? (Enoch Yu)

How many ways are there to choose numbers from 1-999 such that the sum is divisible by 3?

↳ $\{1, 2, \dots, 999\}$ How many subsets of the set have the elements in which their sum is divisible by 3?

$\{3\}$, $\{1,2\}$, $\{999, 981\}$

$$f(x) = (1+x)(1+x^2)\cdots(1+x^{999})$$

1

Exponent: 6, coefficient: 4

$\{6\}$
$\{1,5\}$
$\{2,4\}$
$\{1,2,3\}$

4

$$\sim x^0 + \sim x^3 + \sim x^6 + \cdots + \sim x^{3n}$$

Sum of the coefficients

Root of Unity of 3 $\Rightarrow e^{i\frac{0\pi}{3}},\ e^{i\frac{2\pi}{3}},\ e^{i\frac{4\pi}{3}}$

$\omega_1 \quad \omega_2 \quad \omega_3$

$500 \cdot 999$

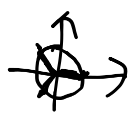
$500000 - 500$

$= 499500$

$$f(x) = \prod_{i=1}^{999}(1+x^i)$$

2

$$= a_0 + a_1x^1 + a_2x^2 + \cdots + a_{499500}x^{499500}$$



$$f(\omega_1) = a_0 + a_1\omega_1 + a_2\omega_1^2 + \cdots + a_{499500}\omega_1^{499500}$$

3

Enoch Yu

$$f(w_2) = a_0 + \cancel{a_1 w_2^1} + \cancel{a_2 w_2^2} + \cdots + a_{999500} w_2^{999500}$$
$$f(w_3) = a_0 + \cancel{a_1 w_3^1} + \cancel{a_2 w_3^2} + \cdots + a_{999500} w_2^{999500}$$
$$+$$

$$f(w_1) + f(w_2) + f(w_3) = 3a_0 + 3a_3 + \cdots + 3a_{999500}$$

$$\frac{f(w_1)+f(w_2)+f(w_3)}{3}$$

$$w_2^2 = w_3$$
$$w_2^3 = w_1$$
$$f(w_2) = \Big(\underbrace{(1+w_1)(1+w_2)(1+w_3)}_{2}\Big)^{333} \quad \leftrightarrow \quad (x-1)^3 = 1$$
$$f(w_2) = 2^{333}$$
$$f(w_3) = 2^{333}$$

$$f(x) = (1+x)(1+x^2)\cdots(1+x^{999})$$
$$f(w_1) = 2^{999}$$
$$f(w_2) = \big((1+w_2)(1+w_2^2)(1+w_2^3)\big)^{333}$$
$$f(w_3) = \big((1+w_3)(1+w_3^2)(1+w_3^3)\big)^{333}$$

$$\frac{2^{999} + 2^{333} + 2^{333}}{3}$$

$$\therefore \frac{2^{334} + 2^{999}}{3}$$