Enoch Yu

Prove that for positive reals $a$, $b$, $c$ we have

$$3(a+b+c) \geq 8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3+b^3+c^3}{3}}.$$

pf)

$?\,(a+b+c) \geq 8\sqrt[3]{abc}$ ✓ ①

$??\,(a+b+c) \geq \sqrt[3]{\frac{a^3+b^3+c^3}{3}}$

$?+??=3$,

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

$$\frac{8}{3}(a+b+c) \geq 8\sqrt[3]{abc}$$ ✓

$$\frac{1}{3}(a+b+c) \geq \sqrt[3]{\frac{a^3+b^3+c^3}{3}}$$

lemma $\frac{a+b+c}{3} \geq \sqrt[3]{\frac{a^3+b^3+c^3}{3}}$ ✓ ②

pf) $\frac{(a+b+c)^3}{27} \geq \frac{a^3+b^3+c^3}{3}$

$$a^3+b^3+c^3+3(a^2b+b^2c+c^2a+ab^2+bc^2+ca^2)+6abc \geq 9a^3+9b^3+9c^3$$

$$\cancel{8a^3}+\cancel{8b^3}+\cancel{8c^3}-\cancel{3a^2b}-\cancel{3b^2c}-\cancel{3c^2a}-\cancel{3ab^2}-\cancel{3bc^2}-\cancel{3ca^2}-\cancel{6abc} \leq 0$$

$$a(8a^2-3b^2-3c^2-2bc)+b(8b^2-3c^2-3a^2-2ca)+c(8c^2-3a^2-3b^2-2ab) \leq 0$$

WLOG, $a \geq b \geq c > 0$

$$a\left(8a^2-2b^2-2c^2-(b+c)^2\right)+b\left(8b^2-2c^2-2a^2-(c+a)^2\right)+c\left(8c^2-2a^2-2b^2-(a+b)^2\right) \leq 0$$

$a=b=c$ //

$8a^2-2b^2-2c^2-(b+c)^2$ ✓ ③

| | | | |
|---|---|---|---|
| ↳ $a\uparrow$ | $b\downarrow$ | $c\downarrow$ | $a,b,c \geq 0$ |
| $a\downarrow$ | $b\uparrow$ | $c\downarrow$ | |
| $a\downarrow$ | $b\downarrow$ | $c\uparrow$ | |

only $a$ is variable

$b, c$ → constants

$a>b\geq c$.

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$0 \leq 0$

$\square$

$\square$