Enoch Yu

Consider the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$. What are the last three digits (from left to right) of the 2020^{th} term?

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$ ①

$F_{2020} \equiv a \pmod{1000}$

$F_{2020} \equiv b \pmod{8}$ ②

$F_{2020} \equiv c \pmod{125}$

$$F_{2020} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{2020} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{2020}$$

$$= \frac{1}{\sqrt{5}} \cdot \frac{1}{2^{2020}}\left((1+\sqrt{5})^{2020} - (1-\sqrt{5})^{2020}\right)$$ ③

$$= \frac{1}{\sqrt{5}} \cdot \frac{1}{2^{2020}}\left(\cancel{\binom{2020}{0}1^{2020}\sqrt{5}^0} + \binom{2020}{1}1^{2019}\sqrt{5}^1 + \cdots + \cancel{\binom{2020}{2020}1^0\sqrt{5}^{2020}}\right.$$

$$\left. - \cancel{\binom{2020}{0}1^{2020}\sqrt{5}^0} + \binom{2020}{1}1^{2019}\sqrt{5}^1 - \cdots - \cancel{\binom{2020}{2020}1^0\sqrt{5}^{2020}}\right)$$

$$= \frac{1}{\sqrt{5}} \cdot \frac{1}{2^{2020}} 2\left(\binom{2020}{1}1^{2019}\sqrt{5}^1 + \binom{2020}{3}1^{2017}\sqrt{5}^3 + \cdots + \binom{2020}{2019}1^1\sqrt{5}^{2019}\right)$$

$$= \frac{1}{2^{2019}}\left(\binom{2020}{1}\sqrt{5}^0 + \binom{2020}{3}\sqrt{5}^2 + \cdots + \binom{2020}{2019}\sqrt{5}^{2018}\right)$$

④

$\underbrace{1\ 1\ 2\ 3\ 5\ 0\ 5\ 5\ 2\ 7\ 1\ 0}_{12}\ |\ 1\ 1\ 2\ 3$

$$\begin{array}{r} 168 \\ 12\,\overline{)2020} \\ \underline{12} \\ 82 \\ \underline{72} \\ 100 \\ \underline{96} \\ 4 \end{array}$$

$2020 \equiv 4 \pmod{12}$

$F_{2020} \equiv 3 \pmod{8}$

$\underline{\underline{b = 3}}$

$$F_{2020} = \frac{1}{2^{2019}}\left(\binom{2020}{1}\sqrt{5}^0 + \binom{2020}{3}\sqrt{5}^2 + \cdots + \binom{2020}{2019}\sqrt{5}^{2018}\right)$$

$F_{2020} \equiv c \pmod{125}$

$$\binom{2020}{1}\sqrt{5}^0 + \binom{2020}{3}\sqrt{5}^2 + \cdots + \binom{2020}{2019}\sqrt{5}^{2018} \equiv 2^{2019}c \pmod{125}$$

$(2, 125) = 1$

$2^{\varphi(125)} \equiv 1 \pmod{125}$

$125 \cdot \frac{4}{5} = 100$

$2^{100} \equiv 1 \pmod{125}$

$2^{2019} \equiv 2^{19} \pmod{125}$ ⑤

$$\equiv 1024 \cdot 512 \pmod{125}$$
$$\equiv 24 \cdot 12 \pmod{125}$$
$$\equiv 144 \cdot 2 \pmod{125}$$
$$\equiv 38 \pmod{125}$$

$$\binom{2020}{1}\sqrt{5}^0 + \cdots + \binom{2020}{2019}\sqrt{5}^{2018} \equiv 38C \pmod{125}$$

$$\binom{2020}{1}\sqrt{5}^0 + \binom{2020}{3}\sqrt{5}^2 + \binom{2020}{5}\sqrt{5}^4 \equiv 38C \pmod{125}$$

$$2020 + \frac{2020 \cdot \cancel{2019} \cdot \cancel{2018}}{\cancel{3} \cdot \cancel{2}} \cdot 5 + \frac{2020 \cdot \cancel{2019} \cdot \cancel{2018} \cdot 2017 \cdot \cancel{2016}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} \cdot 5^2 \equiv 38C \pmod{125}$$

673 1009   673 1009 504

$2020 \cdot 673 \cdot 1009 \cdot 5$: 20, 100, 9, 200, 2100, 200, 45

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$
$$20 \qquad 45 \qquad 100$$

$2020 \cdot 673 \cdot 1009 \cdot 2017 \cdot 504 \cdot 5$: 20, 105, 9, 17, 4, 20, 3600, 105, 300, 5100, 100

145

$$20 + 45 + 100 \equiv 38C \pmod{125}$$
$$38C \equiv 20 \pmod{125}$$

$$38^{-1} \equiv ? \pmod{125}$$

114

$(38, 125) = 1$

$$125 = 3 \cdot 38 + 11$$
$$38 = 3 \cdot 11 + 5$$
$$11 = 2 \cdot 5 + 1$$

6

$$38^{-1} \equiv -23 \equiv 102 \pmod{125}$$

$$\begin{aligned} 1 &= 11 - 2 \cdot 5 \\ &= 11 - 2(38 - 3 \cdot 11) \\ &= 7 \cdot 11 - 2 \cdot 38 \\ &= 7(125 - 3 \cdot 38) - 2 \cdot 38 \\ &= 7 \cdot 125 - 23 \cdot 38 \end{aligned}$$

$$38C \cdot 102 \equiv 20 \cdot 102 \pmod{125}$$

$$\begin{aligned} C &\equiv 20 \cdot 102 \pmod{125} \\ &\equiv 140 \pmod{125} \\ &\equiv 15 \pmod{125} \end{aligned}$$

$$\therefore \underline{\underline{C \equiv 15}} \;//$$

$$F_{2020} \equiv 3 \pmod{8} \qquad F_{2020} = 8k + 3$$

Enoch Yu

$F_{2020} \equiv 15 \pmod{125}$

$8k+3 \equiv 15 \pmod{125}$

$8k \equiv 12 \pmod{125}$

$8k \equiv 512 \pmod{125}$

$k \equiv 64 \pmod{125}$

$\therefore k = 125k' + 64$

$F_{2020} = 1000k' + 512 + 3$

$= 1000k' + 515$

$\therefore F_{2020} \equiv 515 \pmod{1000}$

515