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Let $x_1, x_2, \cdots, x_{2023}$ be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1+x_2+\cdots+x_n)(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n})}$$

is an integer for every $n = 1, 2, \cdots, 2023$. Prove that $a_{2023} \geq 3034$.

$a_1 = \sqrt{x_1 \cdot \frac{1}{x_1}} = 1$

$a_2 = \sqrt{(x_1+x_2)(\frac{1}{x_1}+\frac{1}{x_2})} > 2$ ①

$(x_1+x_2)(\frac{1}{x_1}+\frac{1}{x_2}) \geq (1+1)^2 = 4$

$x_1 \neq x_2 \quad \therefore (x_1+x_2)(\frac{1}{x_1}+\frac{1}{x_2}) > 4$

$2023 \cdot \frac{3}{2} \simeq 3034$

$3034 \simeq 2023 \cdot \frac{1}{2} + 2023 \cdot \frac{2}{2}$ ②

$= 1 + 1011 + 1011 \cdot 2$

$\hookrightarrow a_1 + 1 \cdot 1011 + 2 \cdot 1011 = 3034$

$\underbrace{a_1 \quad a_2}_{+1} \quad \underbrace{a_2 \quad a_3}_{\geq 2} \quad \underbrace{\cdots}_{1} \quad \underbrace{\quad a_n}_{\geq 2}$

$a_{2023} \geq 3034$

lemma If $a_n - a_{n-1} = 1$, then $a_{n+1} - a_n \geq 2$

$a_{2023} = (a_{2023} - a_{2021}) + (a_{2021} - a_{2020}) + \cdots + (a_2 - a_1) + a_1$

$\geq 1 + 1 \cdot 1011 + 2 \cdot 1011 = 3034$

If we prove our lemma, then, $a_{2023} \geq 3034$

lemma If $a_{n+1} - a_n = 1$, then $a_{n+2} - a_{n+1} \geq 2$ ③

prove

$a_{n+1}^2 = (x_1 + \cdots + x_{n+1})(\frac{1}{x_1} + \cdots + \frac{1}{x_{n+1}})$

$= (x_1 + \cdots + x_n)(\frac{1}{x_1} + \cdots + \frac{1}{x_n}) + \frac{1}{x_{n+1}}(x_1 + \cdots + x_n) + x_{n+1}(\frac{1}{x_1} + \cdots + \frac{1}{x_n}) + 1$

$= a_n^2 + \underbrace{\frac{1}{x_{n+1}}(x_1 + \cdots + x_n) + x_{n+1}(\frac{1}{x_1} + \cdots + \frac{1}{x_n})} + 1$

$\geq a_n^2 + 2\sqrt{\cancel{\frac{1}{x_{n+1}}} \cdot \cancel{x_{n+1}} \cdot (x_1 + \cdots + x_n)(\frac{1}{x_1} + \cdots + \frac{1}{x_n})} + 1$ ④

$\geq a_n^2 + 2a_n + 1$

$\geq (a_n + 1)^2$

✓ × $a_{n+2} \geq 2 + a_{n+1}$

× ✓ $a_{n+2} = 1 + a_{n+1}$

$\cancel{a_{n+2} \geq a_{n+1}}$

$a_{n+1}^2 \geq (a_n + 1)^2$

$a_{n+1} = a_n + 1, \quad a_{n+2} - a_{n+1} = 1$

$a_{n+1} = a_n + 1 \iff \frac{1}{x_{n+1}}(x_1 + \cdots + x_n) = x_{n+1}(\frac{1}{x_1} + \cdots + \frac{1}{x_n})$

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$$\frac{1}{x_{n+2}}(x_1+\cdots+x_{n+1}) = x_{n+2}\left(\frac{1}{x_1}+\cdots+\frac{1}{x_{n+1}}\right)$$

$$\frac{1}{x_{n+2}}(x_1+\cdots+x_n)+\frac{x_{n+1}}{x_{n+2}} = x_{n+2}\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)+\frac{x_{n+2}}{x_{n+1}}$$

$$\frac{1}{x_{n+2}}\left((x_1+\cdots+x_n)+x_{n+1}\right) = x_{n+2}\left(\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)+\frac{1}{x_{n+1}}\right)$$

$$\frac{x_{n+1}}{x_{n+2}}\left(\frac{x_1+\cdots+x_n}{x_{n+1}}+1\right) = \frac{x_{n+2}}{x_{n+1}}\left(x_{n+1}\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)+1\right)$$

$$\frac{x_{n+1}}{x_{n+2}} = \frac{x_{n+2}}{x_{n+1}}$$

contradiction

$$\therefore x_{n+1} = x_{n+2}$$

$\square$