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36th KMO II - High Problem 1.

Suppose three sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ satisfy following properties.

- $a_1 = 2,\ b_1 = 4,\ c_1 = 5$
- $a_{n+1} = b_n + \frac{1}{c_n}$, $b_{n+1} = c_n + \frac{1}{a_n}$, and $c_{n+1} = a_n + \frac{1}{b_n}$ are true for all natural numbers n.

For all positive integers n, prove that there exists a number greater than $\sqrt{2n+13}$ from a_n, b_n, and c_n.

D-103 → 100 problem

$a_1 = 2$ $\quad a_{n+1} = b_n + \frac{1}{c_n}$

$b_1 = 4$ $\quad b_{n+1} = c_n + \frac{1}{a_n}$

$c_1 = 5$ $\quad c_{n+1} = a_n + \frac{1}{b_n}$

$$\boxed{a_{n+1} + b_{n+1} + c_{n+1} = a_n + b_n + c_n + \frac{1}{a_n} + \frac{1}{b_n} + \frac{1}{c_n}}$$ ① ↳ symmetric

$\forall n \in \mathbb{N}\ \exists\ a_n, b_n, \text{or } c_n$ s.t. $M_n > \sqrt{2n+13}$ ② ↳ $\boxed{\text{why } \sqrt{\ }?}$ Induction?? ↳ X $\quad a_n, b_n, c_n$ ~~to~~

let. $M_n = \operatorname{Max}(a_n, b_n, c_n)$

$\underline{M_n^2 > 2n+13}$ $\because$ $\underline{M_n > 0}$

$$\underbrace{a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2}_{} = \underbrace{a_n^2 + b_n^2 + c_n^2}_{} + \frac{1}{a_n^2} + \frac{1}{b_n^2} + \frac{1}{c_n^2} + 2\left(\frac{c_n}{a_n} + \frac{a_n}{b_n} + \frac{b_n}{c_n}\right)$$

← AM-GM

$$\frac{\frac{c_n}{a_n} + \frac{a_n}{b_n} + \frac{b_n}{c_n}}{3} \geq \sqrt[3]{\frac{\cancel{c_n \cdot a_n \cdot b_n}}{\cancel{a_n \cdot b_n \cdot c_n}}}$$

$S_n = a_n^2 + b_n^2 + c_n^2$

$S_{n+1} = a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2$

$$\boxed{S_{n+1} > S_n + 6}$$ ③

$$S_{n+1} > S_n + 2\left(\frac{c_n}{a_n} + \frac{a_n}{b_n} + \frac{b_n}{c_n}\right) \geq 6 + S_n$$

$S_1 = 4 + 16 + 25 = 45 = 6 + 39$

$S_1 = 6 + 39$

$S_2 > 39 + 6 + 6$

$S_3 > 39 + 6 + 6 + 6$

$S_4 > 39 + 6 \cdot 4$

$\vdots$ By induction

$$\boxed{S_n > 39 + 6n}$$

$\frac{S_n}{3} > 13 + 2n$

$$M_n^2 \geq \frac{a_n^2 + b_n^2 + c_n^2}{3}$$

← Arithmetic Mean ④

$$\boxed{M_n^2 \geq \frac{S_n}{3} > 2n+13}$$

↳ $M_n^2 > 2n+13$ $\quad \because (M_n > 0)$

$M_n > \sqrt{2n+13}$

$\operatorname{Max}(a_n, b_n, c_n) > \sqrt{2n+13}$

$\exists$ a number from a_n, b_n, c_n that is greater than $\sqrt{2n+13}$

$\square$